Solving by Factorising

Fact — If $pq = 0$ then $p = 0$ or $q = 0$. This is why we factorise: it is the *only* reason a factorised quadratic gives its solutions. (Note $pq = 6$ tells you nothing of the kind.)

Example

Solve:

1. $2x^2 + 3x = 0$
2. $12x^2 - 14x = 6$
3. $4x^2 - 9 = 0$

Quadratics in Disguise

Example

Solve:

1. $x^4 - 13x^2 + 36 = 0$
2. $x - \sqrt{x} - 6 = 0$
3. $2^{2x} - 9 \cdot 2^x + 8 = 0$

Textbook Exercises: SPS Course 2.7, Revision Exercise 2.7.0

Completing the Square

Fact —

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

For a non-monic quadratic, first take out the coefficient of x^2 from the x-terms.

Example

Write in the form $a(x+p)^2 + q$:

1. $x^2 - 6x + 1$
2. $3x^2 - 12x + 5$
3. $2 + 8x - x^2$

Example

By completing the square, solve $2x^2 - 12x + 7 = 0$, giving exact answers.

Textbook Exercises: SPS Course 2.9, Exercises 2 and 3; SPS Course 2.7, Exercises 1A and 1B

The Quadratic Formula

Theorem

$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Derive this by completing the square on $ax^2 + bx + c = 0$.

Example

Solve $2x^2 - 5x + 1 = 0$, giving your answers in exact form.

Remark. All written steps must be shown: quote the formula, substitute, simplify. A calculator answer alone scores nothing.

Textbook Exercises: SPS Course 2.7, Exercise 2

The Discriminant

Definition. The discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$ — the quantity under the square root in the formula.

$b^2 - 4ac > 0$	two distinct real roots
$b^2 - 4ac = 0$	one repeated root (the graph touches the x-axis)
$b^2 - 4ac < 0$	no real roots

Fact (Quadratic inequalities) — To solve $ax^2 + bx + c < 0$ (or > 0): find the roots, sketch the parabola, read off the interval. Never divide an inequality by a variable.

Example

Find the values of k for which $x^2 + 2kx + 25 = 0$ has real roots.

Example

Find the values of p for which $px^2 + px - 6x + 2 = 0$ has no real roots.

Example

The line $y = 2x + c$ is a tangent to the curve $y = x^2 - 4x + 5$. Find c.

Textbook Exercises: SPS Course 2.7, Exercises 3 and 4

The Vertex and Sketching

Fact — $y = a(x+p)^2 + q$ has vertex $(-p, q)$: minimum if $a > 0$, maximum if $a < 0$. The line of symmetry is $x = -p$.

A full sketch shows the shape, the vertex, the y-intercept and any x-intercepts.

Example

Sketch $y = 2x^2 - 8x + 3$, labelling the vertex and all intercepts.

Example

$f(x) = 18 - (x-4)^2,\ x \in \mathbb{R}$.

1. State the maximum value of f and the range of f.
2. The domain is restricted to $x \geq k$ so that f^{-1} exists. State the smallest such k and find f^{-1}.

Example

A quadratic has vertex $(3, -2)$ and passes through $(5, 6)$. Find its equation.

Textbook Exercises: SPS Course 2.7, Revision Exercise 5 and Exam Questions 2.7.8